Final examination 2014
M.Math. II — Commutative algebra
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Throughout R stands for a commutative ring with $1 \neq 0$.

Group A

Answer any two questions from group A

Q 1.
Let R be a ring and $R[X]$ be the polynomial ring.
(a) Let $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. Show that f is nilpotent in $R[X]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent in R.
(b) Show that the Jacobson radical and nilradical coincide for the polynomial ring $R[X]$.

Q 2.
Let M be an R-module and $M_1, M_2, \ldots, M_n$ be submodules of M. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ if and only if
(i) $M = M_1 + M_2 + \cdots + M_n$ and
(ii) $M_i \cap (M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n) = 0$, for $1 \leq i \leq n$.

Q 3.
A multiplicatively closed subset S of the ring R is called saturated if $ab \in S$ implies $a \in S$ and $b \in S$. Show that S is saturated if and only if $R - S$ is union of prime ideals.

Group B

Answer any three questions from group B

Q 4.
Let R be a subring of S and S be integral over R.
(a) Show that if $x \in R$ is a unit in S then it is a unit in R.
(b) Show that the Jacobson radical of R is the contraction of the Jacobson radical of S.

Q 5.
Let $R \subseteq S$ be rings. Let $f(X), g(X)$ be monic polynomials in $S[X]$. Show that if the product $f(X)g(X)$ has coefficients integral over R, then $f(X)$ and $g(X)$ have coefficients integral over R.

Q 6.
Let R be a domain and a valuation ring of its field of fractions K.
(a) Show that R is a local ring.
(b) Show that R is integrally closed in K.

Q 7.
Let R be a domain with K be its field of fractions and $R \neq K$. Prove that the following are equivalent
(i) R is a valuation ring in which every prime ideal is maximal.
(ii) There are no rings properly between R and K.

Group C

Answer any three questions from group C

Q 8.
Show that in a Noetherian ring every irreducible ideal is primary.

Q 9.
Let M be a Noetherian R module. Let $M[X]$ denote the set of all polynomials in X whose coefficients are in M. Show that $M[X]$ is a Noetherian $R[X]$ module.

Q 10.
Show that in an Artin ring the nilradical is nilpotent.

Q 11.
Let k be a field and R be a finitely generated k algebra. Show that the following are equivalent:
(i) R is Artinian,
(ii) R is a finite k algebra.

Group D

Answer all questions from group D

Q 12.
Show that in a Noetherian ring every prime ideal has finite height.

Q 13.
Let $(R, \mathfrak{m})$ be Noetherian local ring. Show that $\dim(R) \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. Dimension on the right hand side is the dimension of $(\mathfrak{m}/\mathfrak{m}^2)$ as a vector space over the residue field $R/\mathfrak{m}$.

Q 14.
Let $R \subset S$ be domains, S be integral over R and R be integrally closed. Let q be a prime ideal of S and $p = q \cap R$. Show that $\dim(R_p) = \dim(S_q)$.